Student Number: ____

NORTHERN BEACHES SECONDARY COLLEGE

MANLY SELECTIVE CAMPUS

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

2015

Mathematics Extension II

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Write your Student Number at the top of each page
- Answer Section I- Multiple Choice on Answer Sheet provided
- Answer Section II – Free Response in a separate booklet for each question.
- Board approved calculators and templates may be used.

Section I Multiple Choice
- 10 marks
- Attempt all questions
- Allow about 15 minutes for this section

Section II – Free Response
- 90 marks
- Each question is of equal value
- All necessary working should be shown in every question.
- Allow about 2 hours 45 minutes for this section

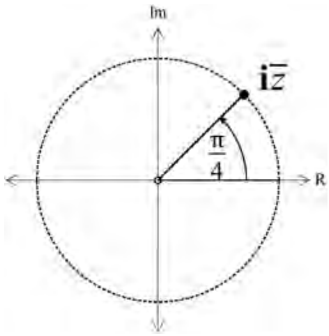
Weighting: 40%

Student Number: ____________________________

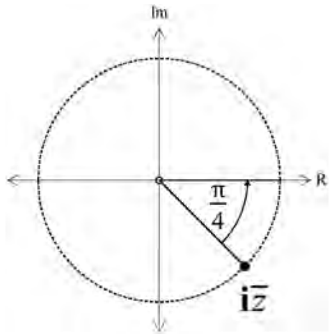
Section 1: Multiple Choice (10 marks)
Indicate your answer on answer sheet provided.

Q1. Given the complex number z has $Arg\, z = \frac{\pi}{4}$, which of the following is a correct representation of $i\bar{z}$?

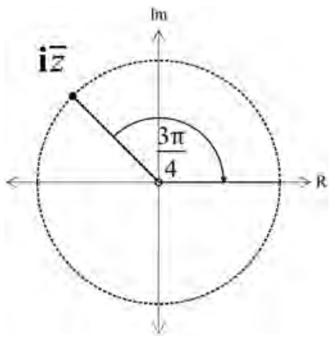
A



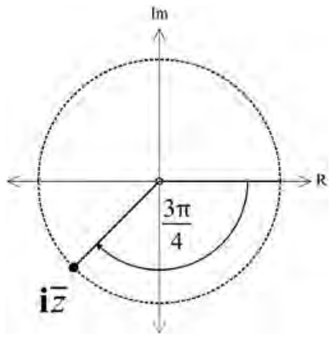
B



C



D



Q2. What is the eccentricity of the hyperbola $4x^2 - 9y^2 = 36$?

A $\frac{\sqrt{13}}{2}$

B $\frac{\sqrt{13}}{3}$

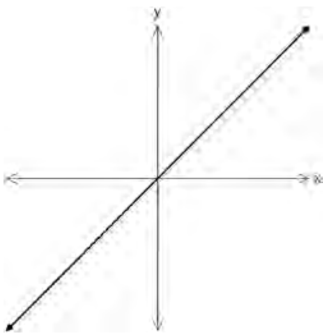
C $\frac{13}{4}$

D $\frac{13}{9}$

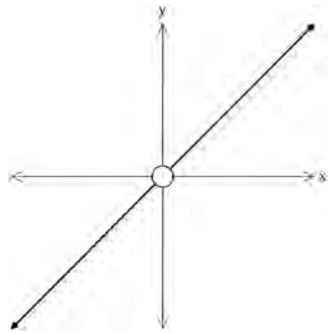
Q3. The equation $\frac{x}{y} + \frac{y}{x} = 2$
defines y implicitly as a function of x.

Which of the following graphs best represents this implicit function?

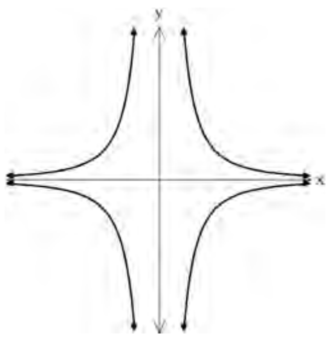
A



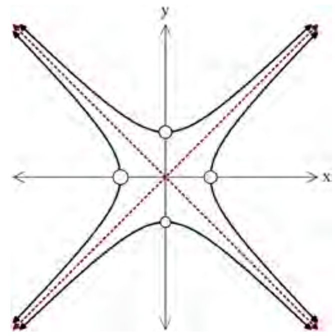
B



C



D



Q4. The polynomial equation $x^3 + 3x^2 - 2x + 6 = 0$ has roots α, β and γ .

Which polynomial has roots $\alpha - 1$, $\beta - 1, \gamma - 1$?

A $x^3 - 5x + 10 = 0$

B $x^3 + 3x^2 - 2x + 14 = 0$

C $x^3 + 6x^2 + x + 8 = 0$

D $x^3 + 6x^2 + 7x + 8 = 0$

Q5. $\int x\sin 2x \, dx =$

A $-\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x + C$

B $-\frac{x}{2}\cos 2x - \frac{1}{4}\sin 2x + C$

C $\frac{x}{2}\cos 2x - \frac{1}{4}\sin 2x + C$

D $-2x\cos 2x + \sin 2x + C$

Q6. If $\int_1^4 f(x)\,dx = 6$, what is the value of $\int_1^4 f(5-x)\,dx =$?

A 6
B 3
C 0
D -6

Q7. The point $P(z)$ moves on the complex plane according to the condition $|z-i|+|z+i|=4$. The Cartesian equation of the locus of P is:

A $\frac{x^2}{4}+\frac{y^2}{3}=1$

B $\frac{x^2}{3}+\frac{y^2}{4}=1$

C $x^2+y^2=1$

D $x^2+y^2=4$

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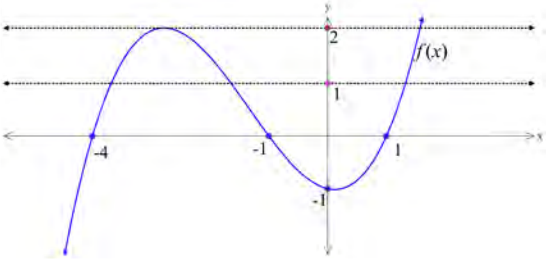
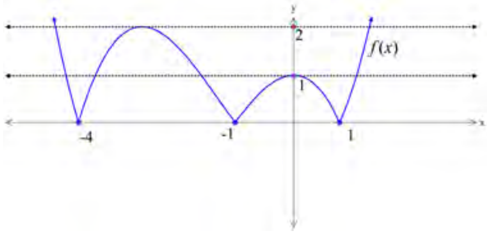
Q8.

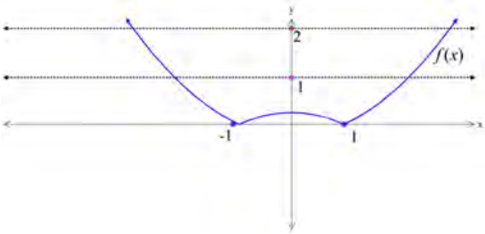


The graph $y = f(x)$ is shown in the figure above. Which of the following could be the graph of $y = f(|x|)$

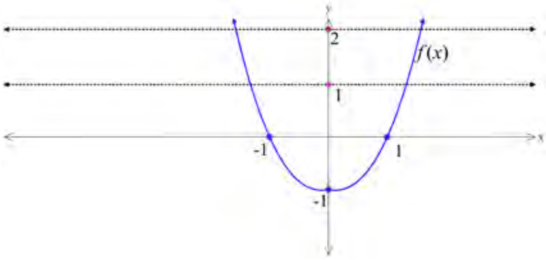
A



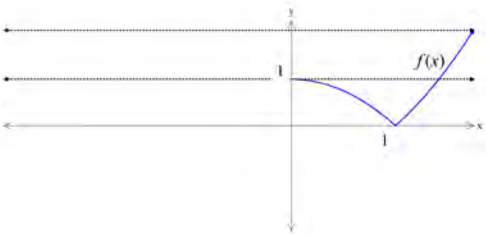
B



C



D



Q9. Let R be the region between the graphs of $y = 1$ and $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$.
The volume of the solid obtained by revolving R around the x – axis is given by

A $2\pi \int_0^{\frac{\pi}{2}} x \sin x \, dx$

B $2\pi \int_0^{\frac{\pi}{2}} x \cos x \, dx$

C $\pi \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

D $\pi \int_0^{\frac{\pi}{2}} (1 - \sin x)^2 \, dx$

Q10. What is the area of the largest rectangle that can be inscribed in the ellipse a
$4x^2 + 9y^2 = 36$?

A $24\sqrt{2}$

B $6\sqrt{2}$

C 24

D 12

End of Multiple Choice

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Section II Total Marks is 90

Attempt Questions 11 – 16.

Allow approximately 2 hours & 45 minutes for this section.

Answer all questions, starting each new question in a new booklet with your **student ID number** in the top right hand corner and the question number on the left hand side of your paper. All necessary working must be shown in each and every question.

Question 11. – Start New Booklet **15 marks**

a) Find $\sqrt{8-6i}$ **(2)**

b) Factorise $z^2 + 2iz + 15$ **(2)**

c) Given the expression $\dfrac{6x^2+3x+1}{(x+1)(x^2+1)}$

(i) Find numbers A, B and C such that

$$\frac{6x^2+3x+1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$ **(2)**

(ii) Find $\displaystyle\int \frac{6x^2+3x+1}{(x+1)(x^2+1)}\,dx$ **(3)**

d) Evaluate $\displaystyle\int_0^{\sqrt{3}} 75x^3\sqrt{1+x^2}\,dx$ **(3)**

e) Given the polynomial $P(x) = x^4 - x^3 + 7x^2 - 9x - 18$

(i) Show that $(x+1)$ is a factor of $P(x)$ **(1)**

(ii) Factorise $P(x)$ over the complex field. **(2)**

Question 12 Start New Booklet **15 Marks**

a) Find the exact value of $\int_{-\sqrt{2}}^{\sqrt{2}} \frac{e^x}{e^x+1}\,dx$. **(3)**

(Give your answer in simplest exact form.)

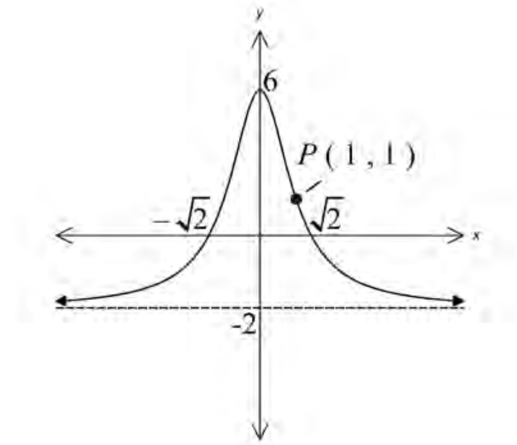
b) The point *P* on the Argand diagram represents the complex number *z*, where *z* satisfies:

$$\frac{2}{z}+\frac{2}{\bar{z}}=1$$

(i) Find the Cartesian equation of the locus of *P* as *z* varies. **(2)**

(ii) Sketch the locus of *P* **(1)**

c) The diagram below shows the graph $y=f(x)$.
The point *P* has the coordinates (1,1).



On separate diagrams sketch the graphs of the following functions.

Each sketch must be at least $\frac{1}{3}$ of a page and must be clearly labelled, showing the coordinate axes, origin and all significant features.

(i) $y=|f(x)|$ **(1)**

(ii) $y=\log\{f(x)\}$ **(1)**

(iii) $y=[f(x)]^2$ **(2)**

(iv) The inverse function $y=f^{-1}(x)$ **(2)**

(v) $y=\frac{1}{6-f(x)}$ **(3)**

Question 13 – Start New Booklet **15 Marks**

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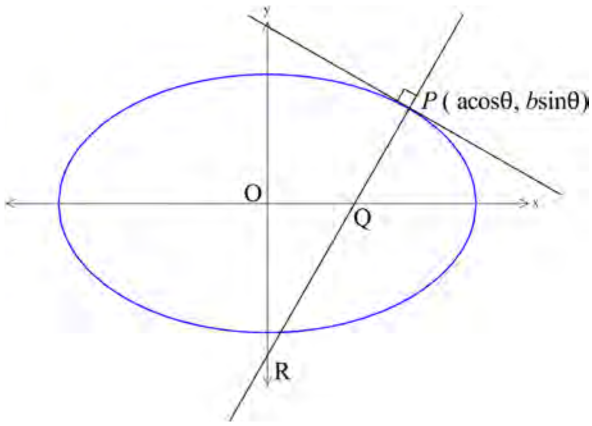
a)



P $(a\cos\theta, b\sin\theta)$, where $0 < \theta < \frac{\pi}{2}$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(i) Show that the equation of the normal at P is

$$a(\sin\theta)x - (b\cos\theta)y = (a^2 - b^2)\sin\theta\cos\theta$$ **(2)**

(ii) Show that $\Delta\ QOR$ has an area given by

$$A = \frac{(a^2 - b^2)^2}{2ab}\sin\theta\cos\theta$$ **(2)**

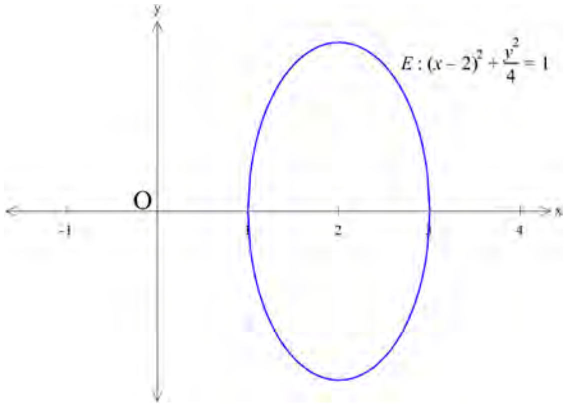
(iii) Find the maximum value of A **(2)**

(iv) Find the coordinates of P for which the maximum value of A occurs. **(1)**

Question 13 continues next page.

Question 13 continued.

b)



The region enclosed by the ellipse $E:(x-2)^2+\frac{y^2}{4}=1$ is rotated through one complete revolution around the y axis.

(i) Use the method of cylindrical shells to show that the volume V of the solid of revolution is given by

$$V = 8\pi\int_1^3 x\sqrt{1-(x-2)^2}\,dx$$ **(2)**

(ii) Hence find the volume of the solid of revolution in its simplest exact form. **(4)**

c) Find the gradient of the tangent to curve $x^2+xy-y^2=11$ at the point $P(3,1)$ **(2)**

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Question 14 – Start New Booklet **15 Marks**

a) Show that $\int \frac{\sin^3 x}{\cos^4 x}\, dx = \frac{1}{3}\sec^3 x - \sec x$ **(3)**

b) (i) Show that $\int_{-a}^{0} f(x)\, dx = \int_{0}^{a} f(-x)\, dx$ **(2)**

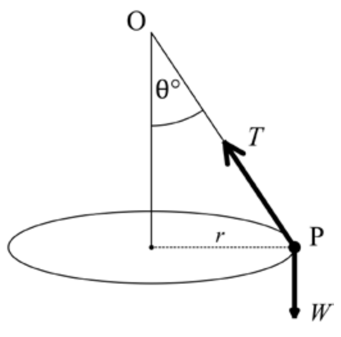
(ii) Deduce that $\int_{-a}^{a} f(x)\, dx = \int_{0}^{a} \{f(x) + f(-x)\}\, dx$ **(1)**

(iii) Hence, or otherwise, evaluate $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{1+\sin x}\, dx$ **(3)**

(Express your answer in simplest exact form)

c) A body P of mass 0.5kg is suspended from a fixed point O by means of a light rod of length 1 metre.

The mass is rotated in a horizontal circle at a constant speed v ms^{-1}. The rod makes an angle Θ with the downward vertical direction as shown in the diagram below.



The tension in the rod is T newtons and the weight of P is W newtons. The radius of the circle is r metres.

Assume $g = 9{\cdot}8\, ms^{-2}$ and $\theta = 30°$

(i) Show that $\tan\theta = \frac{v^2}{rg}$ (3)

(ii) Find the tension T. (1)

(iii) Find the speed v ms^{-1} of P (1)

(iv) Find the period of the motion. (1)

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Question 15 – Start a new booklet

a) (i) By considering $z^9 - 1$ as the difference of two cubes, show that

$$z^9 - 1 = (z - 1)(z^2 + z + 1)(z^6 + z^3 + 1)$$ **(1)**

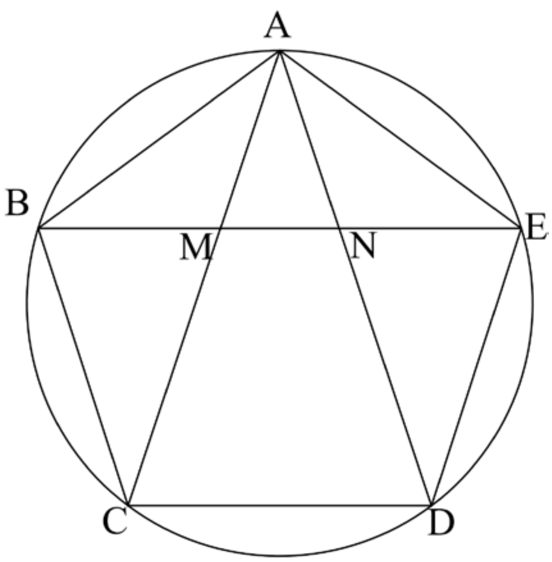
(ii) Use the result $a^n - b^n = (a - b)\left(a^{n-1} + a^{n-2}b + \ldots + b^{n-1}\right)$ to show that

$$z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = (z^2 + z + 1)(z^6 + z^3 + 1)$$ **(1)**

(iii) Solve $z^9 - 1 = 0$ and hence determine the six solutions of $z^6 + z^3 + 1 = 0$ **(2)**

(iv) Hence show that $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$ **(2)**

b)



ABCDE, where *AB* = *AE*, is a pentagon inscribed in a circle. *BE* intersects *AC* and *M* and *AD* at *N* respectively.

(i) Show that $\angle BEA = \angle ACE$ **(2)**

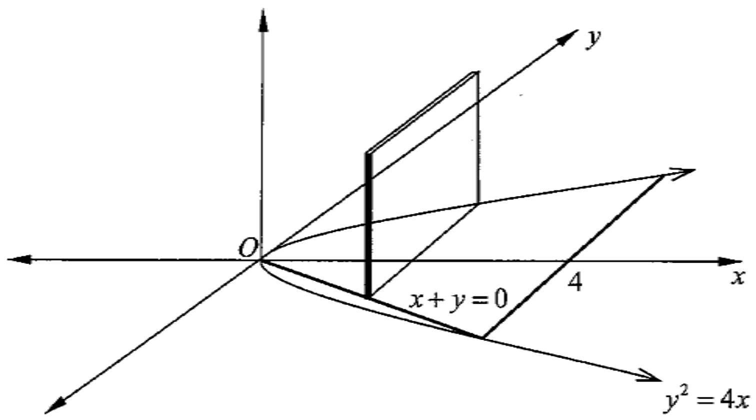
(ii) Hence show that *CDNM* is a cyclic quadrilateral **(3)**

Question 15 – continues on next page.

Question 15 – continued

c) The base of a solid is the region bounded by the curve $y^2 = 4x$ and the lines $x + y = 0$ and $x = 4$.

Every cross section perpendicular to the x axis is a square having a side with one end point on the line $x + y = 0$ and the other on the curve $y^2 = 4x$



(i) Show that the area A of the cross section is given by

$$A = 4x + x^2 + 4x^{\frac{3}{2}}$$

(2)

(ii) Hence find the volume of the solid. (2)

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Question 16 – Start a New Booklet **15 Marks**

a) Let $I_n = \int_0^1 (1-x^2)^n \, dx$ for $n \geq 1$

(i) Show that $I_n = \dfrac{2n}{2n+1} . I_{n-1}$ **(3)**

(ii) Evaluate I_3 **(1)**

b) Consider the rectangular hyperbola $xy = c^2$, where $c > 0$

(i) P and Q are points on the hyperbola with coordinates $(cp , \frac{c}{p})$ and $\left(cq, \dfrac{c}{q}\right)$ respectively.

Prove that the equation of the chord joining P and $Q$is given by:

$$x + pqy = c(p + q)$$ **(2)**

(ii) The chord PQ intersects the x and y axes at M and respectively N respectively.

Prove that $PN = QM$ **(3)**

c) A polynomial $P(x)$ is divided by $x^2 - a^2$ where $a \neq 0$, and the remainder is $px + q$.

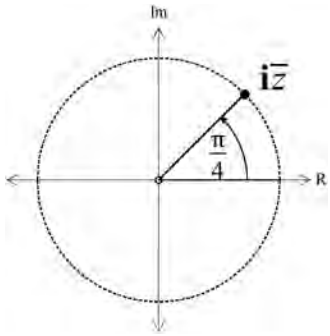
(i) Show that

$p = \dfrac{1}{2a}[P(a) - P(-a)]$ and

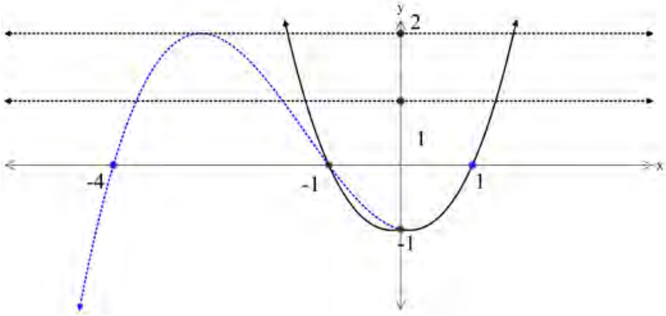
$q = \dfrac{1}{2}[P(a) + P(-a)]$ **(3)**

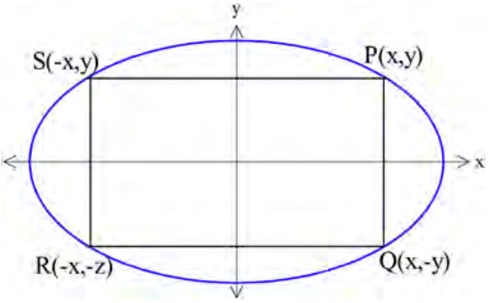
(ii) Find the remainder when $P(x) = x^n - a^n$ for n a positive integer, is divided by $x^2 - a^2$. **(3)**

End of Examination

Q1	 Conjugate of z followed by 90 degree anticlockwise rotation.	A
Q2	$4x^2 - 9y^2 = 36$ $\frac{x^2}{9} - \frac{y^2}{4} = 1 \Rightarrow a^2 = 9 \quad b^2 = 4$ $e^2 = 1 + \frac{b^2}{a^2}$ $e^2 = \frac{13}{9}$ $e = \sqrt{\frac{13}{9}} = \frac{\sqrt{13}}{3}$	B
Q3	$\frac{x}{y} + \frac{y}{x} = 2$ $x^2 + y^2 = 2xy$ $x^2 - 2xy + y^2 = 0$ $(x-y)^2 = 0$ $y = x\ ;\ x \neq 0\ y \neq 0$	B
Q4	$x^3 + 3x^2 - 2x + 6 = 0$ $\alpha - 1\ ;\ \beta - 1\ ;\ \gamma - 1$ $y = x - 1$ $\Rightarrow \quad x = y + 1$ $(y+1)^3 + 3(y+1)^2 - 2(y+1) + 6 = 0$ $y^3 + 3y^2 + 3y + 1 + 3y^2 + 6y + 3 - 2y - 2 + 6 = 0$ $y^3 + 6y^2 + 7y + 5 = 0$ $\Rightarrow \quad x^3 + 6x^2 + 7x + 5 = 0$	D

Q5	$\int x\sin 2x\,dx = -\frac{1}{2}x\cos 2x - \int -\frac{1}{2}(\cos 2x)\,dx$ $u = x \Rightarrow du = 1$ $dv = \sin 2x \Rightarrow v = -\frac{1}{2}\cos 2x$ $\int x\sin 2x\,dx = -\frac{1}{2}x\cos 2x + \frac{1}{2}\frac{1}{2}\sin 2x$ $= -\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x$	A
Q6	$\int_1^4 f(5-x)\,dx$ $u = 5 - x$ $x = 1 \Rightarrow u = 4$ $x = 4 \Rightarrow u = 1$ $dx = -du$ $\int_4^1 f(u)(-du)$ $= -\int_4^1 f(u)du$ $= \int_1^4 f(u)\,\mathrm{du}$ $= \int_1^4 f(x)\,dx = 6$	A
Q7	$\lvert z-i\rvert + \lvert z+i\rvert = 4$ "From PS + PS' $=2b$ " the locus is an ellipse with foci $(0, \pm i)$ $\therefore$ $2b = 4$ $\Rightarrow$ $b^2 = 4$ $\therefore$ $\frac{x^2}{3} + \frac{y^2}{4} = 1$	B

Q8		C
Q9	Using disc method $\delta V = \pi(R^2 - r^2)\delta x$ $= \pi[1^2 - (\sin x)^2]\delta x$ $= \pi(\cos^2 x)\delta x$ $V = \pi\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$	C

Q10	 $4x^2 + 9y^2 = 36$ $\frac{x^2}{9} + \frac{y^2}{4} = 1$ $Area = (2x)(2y) = 4xy$ $A^2 = 16x^2y^2$ $\frac{A^2}{4} = 16x^2\left(\frac{y^2}{4}\right)$ $= 16x^2\left(1 - \frac{x^2}{9}\right)$ $H = \frac{A^2}{4} = 16x^2 - \frac{16x^4}{9}$ $\frac{dH}{dx} = 32x - \frac{64x^3}{90} = 0$ $\Rightarrow 32 - \frac{64x^2}{9} = 0$ $x^2 = \frac{9}{2}$ $\Rightarrow y^2 = 2$ $A^2 = 16\left(\frac{9}{2}\right)^2 = 144$ $A = 12$	D

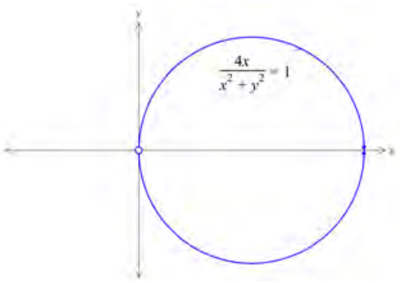
Question 11

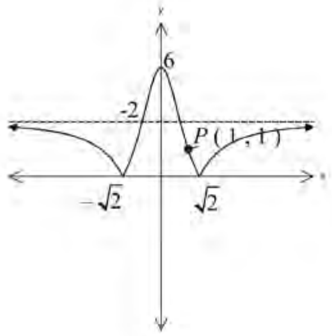
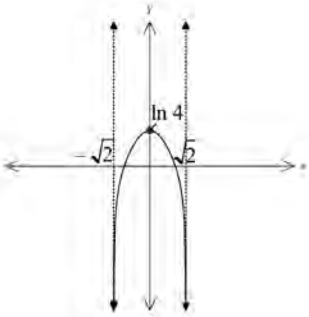
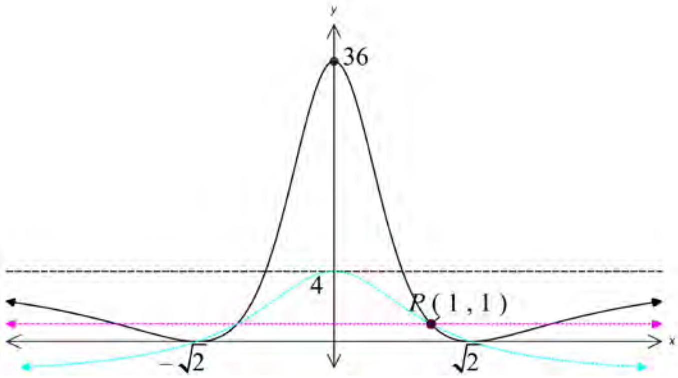
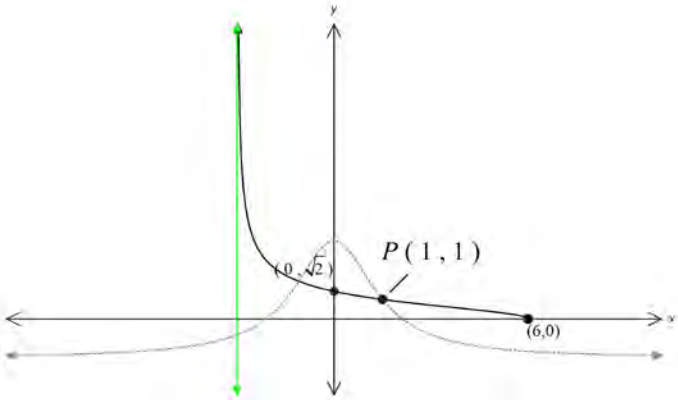
a	$x + iy = \sqrt{8 - 6i}$ $(x + iy)^2 = 8 - 6i$ $x^2 - y^2 = 8 \quad xy = {}^-3 \Rightarrow y = -\frac{3}{x}$ $\therefore \quad x^2 - \frac{9}{x^2} = 8$ $x^4 - 8x^2 - 9 = 0$ $(x^2 - 9)(x^2 + 1) = 0$ $x^2 - 9 = 0 \quad x^2 + 1 = 0$ $x = \pm 3$ No solution $y = \frac{3}{x} \Rightarrow y = \pm 1$ $\sqrt{8 - 6i} = \pm(3 - i)$	2 marks – correct solution 1 mark – obtains equations $x^2 - y^2 = 8$ $xy = {}^-3$
b	$z^2 + 2iz + 15 = 0$ $(z + 5i)(z - 3i) = 0$	2 marks – correct solution 1 mark – solves equation to obtain $z = -5i$, $3i$
c-i	$\frac{6x^2 + 3x + 1}{(x + 1)(x^2 + 1)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1}$ $(6x^2 + 3x + 1) = A(x^2 + 1) + (Bx + C)(x + 1)$ $= (A + B)x^2 + (B + C)x + A + C$ $A + B = 6$ ① $B + C = 3$ ② $A + C = 1$ ③ ③ – ② $A - B = {}^-2$ ④ ① + ④ $2A = 4$ $\Rightarrow A = 2$ $B = 4$ $C = {}^-1$	2 marks – correct solution 1 mark – attempts to equate coefficients from LHS and RHS

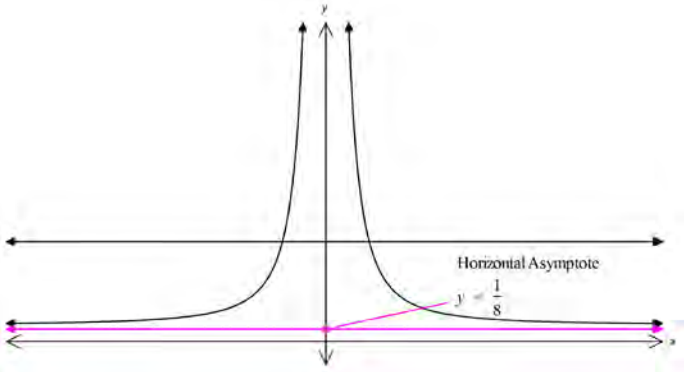
c-ii	$\int \frac{6x^2+3x+1}{(x+1)(x^2+1)}\,dx$ $= \int \left\{\frac{2}{x+1} + \frac{4x-1}{x^2+1}\right\} dx$ $= \int \frac{2}{x+1}\,dx + \int \frac{4x-1}{x^2+1}\,dx$ $= \int \frac{2}{x+1}\,dx + \int \frac{4x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx$ $= 2\ln(x+1) + 2\ln(x^2+1) - \tan^{-1}x + K$	3 marks – correct 2 marks – two correct terms from values of A, B and C 1 mark – obtains one correct term from values of A, B and C
d	Option 1 $\int_0^{\sqrt{3}} 75x^3\sqrt{1+x^2}\,dx$ $u = 1+x^2 \Rightarrow du = 2x\,dx$ $dx = \frac{du}{2x}$ $x^2 = u-1$ $\Rightarrow x = 0 \qquad u = 1$ $x = \sqrt{3} \quad u = 4$ $\int_1^4 75x^3\sqrt{u}\frac{du}{2x} = \frac{75}{2}\int_1^4 x^2\sqrt{u}\ \text{du}$ $= \frac{75}{2}\int_1^4 (u-1)\sqrt{u}\ \text{du}$ $= \frac{75}{2}\int_1^4 u^{\frac{3}{2}} - u^{\frac{1}{2}}\ \text{du}$ $= \frac{75}{2}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^4$ $= \frac{75}{2}\left[\left(\frac{2}{5}4^{\frac{5}{2}} - \frac{2}{3}4^{\frac{3}{2}}\right) - \left(\frac{2}{5}\times 1\right) - \frac{2}{3}\times 1\right)\right]$ $= 290$	3 marks – correct solution 2 marks – correct transformation of integrand and limits of integration 1 mark – correct transformation of integrand OR limits of integration NOTE: Several alternate techniques available to solve this question. Solution using $x = \tan\Theta$ not included here.

d	Option 2 $\int_0^{\sqrt{3}} 75x^3\sqrt{1+x^2}\ dx$ $= 75\int_0^{\sqrt{3}} x \times x^2\sqrt{1+x^2}\ dx$ $= 75\int_0^{\sqrt{3}} x \times \left[(1+x^2)-1\right]\sqrt{1+x^2}\ dx$ $= \frac{75}{2}\left\{\int_0^{\sqrt{3}} 2x \times (1+x^2)^{\frac{3}{2}} - 2x(1+x^2)^{\frac{1}{2}}\ dx\right\}$ $= \frac{75}{2}\left[\frac{2}{5}(1+x^2)^{\frac{5}{2}} - \frac{2}{3}(1+x^2)^{\frac{3}{2}}\right]_0^{\sqrt{3}}$ = see above	
e-i	$P(x) = x^4 - x^3 + 7x^2 - 9x - 18$ $P(^-1) = (^-1)^4 - (^-1)^3 + 7(^-1)^2 - 9(^-1) - 18 = 0$ $\therefore (x+1)$ is a factor	1 mark – correct substitution and evaluation
e-ii	$P(2) = 0 \therefore (x-2)$ is a factor $\therefore (x+1)(x-2) = (x^2 - x - 2)$ is a factor *By* long division or comparing coefficients $P(x) = (x^2 - x - 2)(x^2 + 9)$ $= (x+1)(x-2)(x+3i)(x-3i)$	2 marks – correct solution 1 mark – determines (x-2) is a factor

Q12

a	$\int_{-\sqrt{2}}^{\sqrt{2}} \frac{e^x}{e^x+1}\,dx$ $= \left[\ln(e^x+1)\right]_{-\sqrt{2},\sqrt{2}}$ $= \ln\left(\frac{e^{\sqrt{2}}+1}{e^{-\sqrt{2}}+1}\right)$ $= \ln\left(\frac{e^{\sqrt{2}}+1}{\frac{1+e^{\sqrt{2}}}{e^{\sqrt{2}}}}\right)$ $= \ln\left(e^{\sqrt{2}}\right)$ $= \sqrt{2}$	3 marks – correct solution 2 marks – correct primitive function and simplification to $\ln\left\{\frac{(e\sqrt{2})+1}{e^{-\sqrt{2}}+1}\right\}$ 1 mark – correct primitive function
b-i	$\frac{2}{z}+\frac{2}{\bar{z}}=1$ $z=x+iy \Rightarrow \bar{z}=x-iy$ $\frac{2}{x+iy}+2(x-iy)=1$ $\frac{2x+2iy+2x-2iy}{x^2+y^2}=1$ $4x=x^2+y^2$ $x^2-4x+y^2=0$ $(x-2)^2+y^2=4$ Circle with centre (2,0) and radius 2 except (0,0) not included.	2 marks – correct solution origin need not be excluded 1 mark – obtains $x^2+y^2=4x$ or equivalent expression.
b-ii		1 mark – correct solution – origin must be excluded.

c-i		1 mark – correct solution
c-ii		1 mark – correct solution
c-iii		2 marks – correct solution (must show horizontal asymptote and *y* intercept) 1 mark – correct asymptote or *y* intercept.
c-iv		2 marks – correct solution (Must have *x*, *y* axes and vertical asymptote) 1 mark attempts to reflect *y* = *f(x)* in *y* = *x* with a correct intercept or vertical asymptote.

c-v		3 marks – correct solution (must show correct horizontal asymptote 2 marks – correct shape with a horizontal asymptote 1 mark – correct shape.

Q13

a-i	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\frac{d}{dx}\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}\right\}=\frac{d}{dx}\{1\}$ $\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$ $\frac{dy}{dx}=\frac{-b^2x}{a^2y}$ At $P\ (acos\theta, b\sin\theta)$ $m_1=\frac{-b^2acos\theta}{a^2b\sin\theta}=\frac{-b\cos\theta}{a\sin\theta}$ For Normal $m_2=\frac{a\sin\theta}{b\cos\theta}$ $y-y_1=\frac{a\sin\theta}{b\cos\theta}(x-x_1)$ $y\ b\cos\theta-b^2\sin\theta\cos\theta=x\ a\sin\theta-a^2\sin\theta\cos\theta$ $(a\sin\theta)x-(b\cos\theta)y=(a^2-b^2)\sin\theta\cos\theta$	2 marks correct solution 1 mark – obtains a correct expression for dy/dx.

a-ii	At Q $y = 0$ $\Rightarrow \quad (a\sin\theta)x = (a^2 - b^2)\sin\theta\cos\theta$ $x = \dfrac{(a^2 - b^2)\cos\theta}{a}$ At R $x = 0$ $\Rightarrow \quad (-b\cos\theta)y = (a^2 - b^2)\sin\theta\cos\theta$ $y = \dfrac{-(a^2 - b^2)\sin\theta}{b}$ $\text{Area} = \dfrac{1}{2}(OQ)(OR) \qquad (OR > 0)$ $= \dfrac{1}{2}\dfrac{(a^2 - b^2)\cos\theta}{a} \times \dfrac{(a^2 - b^2)\sin\theta}{b}$ $= \dfrac{(a^2 - b^2)^2\sin\theta\cos\theta}{2ab}$	2 marks – correct solution 1 mark – obtains a correct expression for either x or y intercept.
a-iii	$\text{Area} = \dfrac{(a^2 - b^2)^2\sin\theta\cos\theta}{2ab}$ $= \dfrac{(a^2 - b^2)^2\sin 2\theta}{4ab}$ Maximum value of $\sin 2\theta = 1$ Max. Value of Area $\Rightarrow$ $= \dfrac{(a^2 - b^2)^2}{4ab}$	2 marks – correct solution 1 mark – transforms $\dfrac{1}{2}\sin\theta\cos\theta$ to $\sin(2\theta)$ - obtains a correct expression for $\dfrac{dA}{d\theta}$
a-iv	Max Value of sin 2th occurs at 2th=pi/2 $\Rightarrow \quad \theta = \dfrac{\pi}{4}$ $P(a\cos\theta, b\sin\theta) = P\left(a\cos\dfrac{\pi}{4}, b\sin\dfrac{\pi}{4}\right)$ $= P\left(\dfrac{a}{\sqrt{2}}, \dfrac{b}{\sqrt{2}}\right)$	1 mark – correct solution

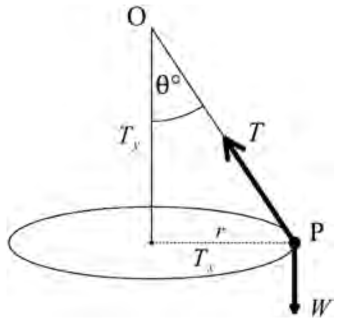
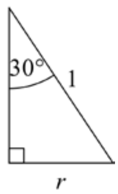
b-i	$\delta V = \text{Circumference} \times \text{Height} \times \text{Thickness}$ $= 2\pi x \times 2y \times \delta x$ $= 4\pi xy\delta x$ $V = \lim_{\delta x \to 0} \sum_{1}^{3} 4\pi xy\delta x$ $= \int_1^3 4\pi xy\ dx$ $(x-2)^2 + \frac{y^2}{4} = 1$ $y^2 = 4[1-(x-2)^2]$ $y = 2\sqrt{1-(x-2)^2}$ $V = \int_1^3 4\pi x\ 2\sqrt{1-(x-2)^2}\ dx$ $= 8\pi \int_1^3 x\ \sqrt{1-(x-2)^2}\ dx$	2 marks – correct solution 1 mark – forms a correct expression for δV
b-ii	Let $u = x-2 \Rightarrow dx = du$ $\therefore\ x = u+2$ $x = 1 \Rightarrow u = {}^-1$ $x = 3 \Rightarrow u = 1$ $V = 8\pi \int_{-1}^{1} (u+2)\sqrt{1-u^2}\ \text{du}$ $= 8\pi \int_{-1}^{1} u\sqrt{1-u^2}\ du + 16\pi \int_{-1}^{1} \sqrt{1-u^2}\ \text{du}$ But $f(u) = u\sqrt{1-u^2}$ $f(-u) = (-u)\sqrt{1-(-u)^2} = -\left(u\sqrt{1-u^2}\right)$ $\therefore\ f(u) = f(-u) \therefore$ odd $\therefore \int_{-a}^{a}$ odd function $= 0$ $\therefore\ V = 8\pi \times 0 + 16\pi \int_{-1}^{1} \sqrt{1-u^2}\ \text{du}$ $= 16\pi \times$ *Area* of semicrcle with radius 1 $= 16\pi \times \frac{1}{2\pi} 1^2$ $= 8\pi^2\ u^3$	4 marks – correct solution 3 marks – forms a correct integral expression and evaluates one term correctly. 2 marks – forms correct integral expression with correct limits 1 mark – changes limits correctly Note: Trig substitution solution also accepted – not shown here.

c	$$x^2 + xy - y^2 = 11$$ $$\frac{d}{dx}\{x^2 + xy - y^2\} = \frac{d(11)}{dx}$$ $$2x + \frac{xdy}{dx} + \frac{ydx}{dx} - \frac{2ydy}{dx} = 0$$ $$x\frac{dy}{dx} - 2y\frac{dy}{dx} = -2x - y$$ $$\frac{dy}{dx}(x - 2y) = -2x - y$$ $$\frac{dy}{dx} = \frac{2x + y}{2y - x}$$ At $P(3,1)$ $$m = \frac{2 \times 3 + 1}{2 \times 1 - 3} = -\frac{7}{1}$$	2 marks – correct solution 1 mark – attempts to use product rule to find dy/dx

Question 14

a	$\int \frac{\sin^3 x}{\cos^4 x}\,dx = \frac{1}{3}\sec^3 x - \sec x$ $u = \cos x$ $du = -\sin x\,dx$ $dx = \frac{du}{-\sin x}$ $I = \int \frac{\sin^3 x}{u^4} - \frac{du}{\sin x}$ $= \int \frac{\sin^2 x}{u^4} du$ $= \int \frac{1-\cos^2 x}{y^4} du$ $= \int \frac{1-u^2}{u^4} du$ $= \int u^{-4} - u^{-2} du$ $= \frac{u^{-1}}{-1} - \frac{u^{-3}}{-3}$ $= -\frac{1}{u} + \frac{1}{3u^3}$ $= -\frac{1}{\cos x} + \frac{1}{3\cos^3 x}$ $= \frac{1}{3}\sec^3 x - \sec x$	3 marks – correct solution 2 marks – forms correct transformed integral 1 mark – substitutes for cos*x* and obtains $\int \frac{\sin^2 x}{u^4}$ du
b-i	$\int_{-a}^{0} f(x)\,dx$ $x = -u \Rightarrow dx = -du$ $x = -a \Rightarrow u = a$ $x = 0 \Rightarrow u = 0$ $\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-u)(-du)$ $= -\int_{a}^{0} f(-u)du$ $= \int_{0}^{a} f(-u)du$ $= \int_{0}^{a} f(-x)dx$	2 marks – correct solution 1 mark – obtains $\int_{a}^{0} f(-u)(-du)$

b-ii	$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$ $= \int_{0}^{a} f(-x)dx + \int_{0}^{a} f(x)dx$ $= \int_{0}^{a} \{f(-x)dx + f(x)\}dx$	1 mark – correct solution
b-iii	$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{1+\sin x}dx$ $= \int_{0}^{\frac{\pi}{6}} \left\{\frac{1}{1+\sin(-x)} + \frac{1}{1+\sin x}\right\} dx$ $= \int_{0}^{\frac{\pi}{6}} \left\{\frac{1}{1-\sin x} + \frac{1}{1+\sin x}\right\} dx$ $= \int_{0}^{\frac{\pi}{6}} \left\{\frac{1+\sin x + 1 - \sin x}{1-\sin^2 x}\right\} dx$ $= \int_{0}^{\frac{\pi}{6}} \left\{\frac{2}{\cos^2 x}\right\} dx$ $= 2\int_{0}^{\frac{\pi}{6}} \sec^2 x dx$ $= 2\left[\tan x\right]_0^{\frac{\pi}{6}}$ $= 2\left\{\tan\frac{\pi}{6} - \tan^0\right\}$ $= 2\left\{\frac{1}{\sqrt{3}} - 0\right\}$ $= \frac{2\sqrt{3}}{3}$	3 marks – correct solution 2 marks – forms a correct expression 1 mark – applies result in b(ii) to $f(x) = \frac{1}{1+\sin x}$

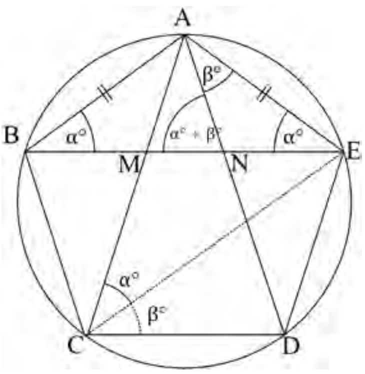
c-i	 $T_x = T\sin\theta$ $T_y = T\cos\theta = W = mg$ $\therefore \quad T = \dfrac{mg}{\cos\theta}$ ① $T = F_C$ $T\sin\theta = \dfrac{mv^2}{r}$ ② Substitute for T $\dfrac{mg}{\cos\theta}\sin\theta = \dfrac{mv^2}{r}$ $g\tan\theta = \dfrac{v^2}{r}$ $\tan\theta = \dfrac{v^3}{rg}$	3 marks – correct solution 2 marks – obtains equations (1) and (2). 1 marks – obtains equations (1) or (2).
c-ii	$T = \dfrac{mg}{\cos\theta}$ $T = \dfrac{0{\cdot}5 \times 9{\cdot}8}{\cos 30^\circ} = 5{\cdot}66\,N$	1 mark – correct solution
c-iii	 $\sin 30^\circ = \dfrac{r}{1} \Rightarrow r = \dfrac{1}{2}$ $v^2 = rg\tan\theta$ $v^2 = \dfrac{1}{2} \times 9{\cdot}8 \times \tan 30^\circ$ $v = \sqrt{2{\cdot}82901}$ $v = 1{\cdot}682\,ms^{-1}$	1 mark – correct solution

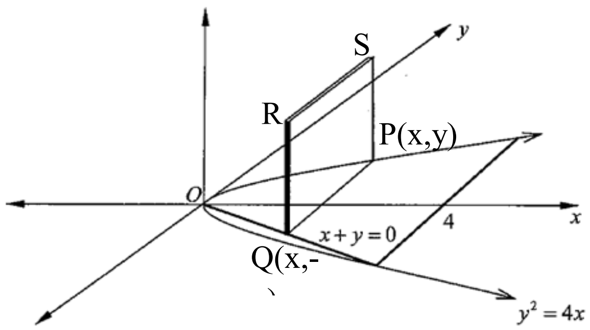
c-iv	$\text{Period} = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{\frac{v}{r}} = \dfrac{2\pi v}{r}$ $= \dfrac{2\pi \times 0{\cdot}5}{1{\cdot}682}$ $= 1{\cdot}9\,s$	1 mark – correct solution

Question 15

a-i	$z^9 - 1 = (z^3)^3 - (1)^3$ $= (z^3 - 1)\left((z^3)^2 + z^3 \times 1 + 1^3\right)$ $= (z^3 - 1)(z^6 + z^3 + 1)$ $= (z - 1)(z^2 + z + 1)(z^6 + z^3 + 1)$	1 mark – correct solution
a-ii	$a^n - b^n = (a - b)\left(a^{n-1} + a^{n-2}b + \ldots + b^{n-1}\right)$ $\Rightarrow \quad z^9 - 1 = (z - 1)(z^8 + z^7 + \ldots + 1)$ $(z - 1)(z^8 + z^7 + \ldots + 1) = (z - 1)(z^2 + z + 1)(z^6 + z^3 + 1)$ $(z^8 + z^7 + \ldots + 1) = (z^2 + z + 1)(z^6 + z^3 + 1)$	1 mark – correct solution

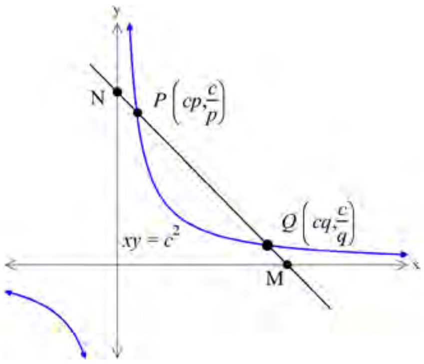
a-iii	$z^9 - 1 = 0 \Rightarrow z^9 = 1$ Let $z = \text{cis}\theta$ $z^9 = (\text{cis}\theta)^9 = \text{cis}(9\theta)$ $\cos(9\theta) + i\sin(9\theta) = 1$ $\therefore \quad \cos(9\theta) = 1$ $9\theta = 2n\pi$ $\theta = \frac{2n\pi}{9} \quad n = \pm(0,1,2,3..)$ $z = \text{cis}\frac{2n\pi}{9}$ $z = 1, \text{cis}\left(\pm\frac{2\pi}{9}\right), \text{cis}\left(\pm\frac{4\pi}{9}\right) \text{cis}\left(\pm\frac{6\pi}{9}\right), \text{cis}\left(\pm\frac{8\pi}{9}\right)$ But solutions of $z^3 - 1 = 0$ $z = 1, \text{cis}\left(\pm\frac{6\pi}{9}\right) = \text{cis}\left(\pm\frac{2\pi}{9}\right)$ $\therefore$ Solutions of $z^6 + z^3 + 1 = 0$ $z = \text{cis}\left(\pm\frac{2\pi}{9}\right), \text{cis}\left(\pm\frac{4\pi}{9}\right), \text{cis}\left(\pm\frac{8\pi}{9}\right)$	2 marks – correct solution 1 mark – obtains $z = \text{cis}\,\frac{2\pi}{9}$
a-iv	$z^6 + z^3 + 1 = 0$ Sum of roots $= -\frac{b}{a} = 0$ $\text{cis}\left(\frac{2\pi}{9}\right) + \text{cis}\left(-\frac{2\pi}{9}\right) + \text{cis}\left(\frac{4\pi}{9}\right) + \text{cis}\left(-\frac{4\pi}{9}\right) + \text{cis}\left(\frac{8\pi}{9}\right) + \text{cis}\left(-\frac{8\pi}{9}\right) = 0 + 0i$ $\cos\left(\frac{2\pi}{9}\right) + \cos\left(-\frac{2\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right) + \cos\left(-\frac{4\pi}{9}\right) + c^0s\left(\frac{8\pi}{9}\right) + \cos\left(-\frac{8\pi}{9}\right) = 0$ nb $\quad \cos\theta = \cos(-\theta)$ $2\cos\left(\frac{2\pi}{9}\right) + 2c^0s\left(\frac{4\pi}{9}\right) + 2\cos\left(\frac{8\pi}{9}\right) = 0$ $\cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right) + \cos\left(\frac{8\pi}{9}\right) = 0$ $\cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right) = -\cos\left(\frac{8\pi}{9}\right)$ $\cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right) = -\cos\left(\pi - \frac{1\pi}{9}\right)$ $\cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right) = -\left\{\cos\pi\cos\frac{\pi}{9} - \sin\pi\sin\frac{\pi}{9}\right\}$ $\cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right) = -\left\{-\cos\frac{\pi}{9}\right\}$ $\cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right) = \cos\frac{\pi}{9}$	2 marks – correct solution 1 mark – uses sum of roots to form an equation

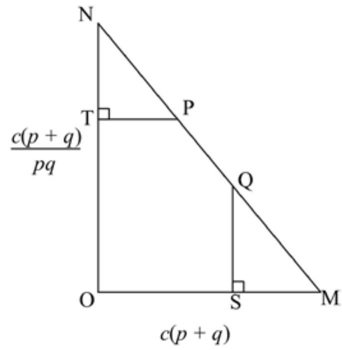
b-i	 Let $\angle BEA = \alpha$ $AB = AE$ given $\therefore$ ΔABE is isosceles $\angle ABE = \alpha$ (base angles of isos triangl) $\angle ACE = \angle BEA = \alpha$ (= $\angle$'s on chordAE)	2 marks correct solution 1 mark - $\angle ABE = \alpha$ with reason or equivalent.
b-ii	Let $\angle EAN = \beta$ $\angle ECD = \beta$ (equal$\angle$'s on chord ED) $\angle ACD = \alpha + \beta$ $\angle ANM = \alpha + \beta$ ($ext\ \angle$ of Δ) $= \angle ACD$ $\therefore CDNM$ is cyclic quad ($ext\ \angle$ = opp interior $\angle$)	3 marks – correct solution 2 marks – correct use of 2 circle geometry rules applicable to question 1 mark - correct use of 1 circle geometry rule applicable to question

c-i	 $PQ = x + y$ $\therefore\ SP = x + y$ Area = $(y + x)^2$ $= y^2 + 2xy + x^2$ But $y^2 = 4x$ $y = \sqrt{4x} = 2\sqrt{x}$ $\therefore\ A = 4x + 2x\,2\sqrt{x} + x^2$ $= 4x + x^2 + 4x^{\frac{3}{2}}$	2 marks – correct solution 1 mark – forms PQ (or SP) $= x + y$
c-ii	$\delta V =$ area $\times\ \delta x$ $V = \lim_{\delta x \to 0} \sum_{x=0}^{4} \left\{4x + x^2 + 4x^{\frac{3}{2}}\right\} \delta x$ $= \int_0^4 \left\{4x + x^2 + 4x^{\frac{3}{2}}\right\} dx$ $V = \left[2x^2 + \frac{x^3}{3} + \frac{8x^{\frac{5}{2}}}{5}\right]_{,0}^{4}$ $= \frac{1568}{15} u^3$	2 marks – correct solution including development of integral 1 mark – correct primitive function or correct evaluation from (equivalent) incorrect primitive.

Question 16

a-i	$I_n = \int_0^1 (1-x^2)^n \, dx$ $u = (1-x^2)^n \quad dv = 1$ $du = {}^-2nx(1-x^2)^{n-1} \quad v = x$ $I_n = \left[x(1-x^2)^n\right](0,1) - \int_0^1 -2nx^2(1-x^2)^{n-1} \, dx$ $= 0 - 2n\int_0^1 -x^2(1-x^2)^{n-1} \, dx$ $= {}^-2n\int_0^1 \{(1-x^2)-1\}(1-x^2)^{n-1} \, dx$ $= {}^-2n\int_0^1 (1-x^2)^n - (1-x^2)^{n-1} \, dx$ $= {}^-2n\int_0^1 (1-x^2)^n \, dx + 2n\int_0^1 (1-x^2)^{n-1} \, dx$ $I_n = {}^-2nI_n + 2nI_{n-1}$ $2nI_n + I_n = 2nI_{n-1}$ $(2n+1)I_n = 2nI_{n-1}$ $I_n = \dfrac{2nI_{n-1}}{2n+1}$	3 marks – correct solution 2 marks – rewrites initial expression for I_n in terms of integrals for I_n and I_{n-1} 1 mark – applies IBP to obtain integral expression to I_n and I_{n-1}

a-ii	$I_3 = \frac{6}{7}I_2 = \frac{6}{7} \times \frac{4}{5}I_1$ $I_1 = \int_0^1 (1 - x^2)dx$ $= \left[x - \frac{x^3}{3}\right]^1, 0 = \frac{2}{3}$ $I_3 = \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} = \frac{16}{35}$	1 mark – correct solution
b-i	 $m_{PQ} = \frac{\frac{c}{p} - \frac{c}{q}}{cp - cq}$ $= \frac{c\left(\frac{q-p}{pq}\right)}{c(p-q)}$ $= -\frac{1}{pq}$ $y - y_1 = -\frac{1}{pq}(x - x_1)$ $y - \frac{c}{p} = -\frac{1}{pq}(x - cp)$ $pqy - cp = -x + cp$ $x + pqy = c(p + q)$	2 marks – correct solution 1 mark – finds correct simplified gradient.

b-ii	 At $M, y = 0 \Rightarrow x = c(p+q)$ At $N, x = 0 \Rightarrow y = \frac{c(p+q)}{pq}$ $\therefore y = \frac{cp}{pq} + \frac{cq}{pq}$ $= \frac{c}{q} + \frac{c}{p}$ $\therefore \quad NR = \frac{c}{q} + \frac{c}{p} - \frac{c}{p} = \frac{c}{q}$ In $\Delta NRP, \Delta QSM$ $\angle RNP = \angle SQM$ (corr $\angle$'s on $\parallel$ lines) $\angle NRP = \angle QSM = 90°$ given/ construct $NR = QS = \frac{c}{q}$ $\Delta NRP = \Delta QSM$ (AAS) $PN \equiv QM$ (corresp sides in cong Δ)	3 marks – correct answer 2 marks – finds NR 1 mark – some
c-i	$P(x) = (x^2 - a^2)Q(x) + px + p$ $= (x-a)(x+a)Q(x) + px + p$ $\therefore \quad P(a) = pa + q$ ① $P(-a) = -pa + q$ ② ① – ② $P(a) - P(-a) = pa - (-pa) = 2pa$ $p = \frac{1}{2a}\{P(a) - P(-a)\}$ ① + ② $P(a) + P(-a) = pa + (-pa) = 2q$ $q = \frac{1}{2}\{P(a) + P(-a)\}$	3 marks – correct solution 2 marks – obtains correct expression s for *P(a)* and *P(-a)*. 1 mark – obtains a correct expression for either *P(a)* or *P(-a)*.

c-ii	$P(x) = x^n - a^n$ If n is even $P(a) = a^n - a^n = 0$ $P(-a) = (-a)^n - a^n = a^n - a^n = 0$ $\therefore$ from part i $p = 0, q = 0$ $\therefore$ no remainder If n is odd $P(a) = a^n - a^n = 0$ $P(-a) = (-a)^n - a^n = -a^n - a^n = {}^-2a^n$ $\Rightarrow \quad p = \frac{1}{2a}\left\{0 - ({}^-2a^n)\right\} = a^{n-1}$ $q = \frac{1}{2}\left\{0 + ({}^-2a^n)\right\} = -a^n$ $R(x) = px + q = a^{n-1}x - a^n$	3 marks – correct solution 2 marks – distinguishes two cases and solves one case correctly. 1 mark – correctly considers 1 case only.